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CYCLIC PURITY VERSUS PURITY IN EXCELLENT NOETHERIAN RINGS

BY

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ABSTRACT. A characterization is given of those Noetherian rings R such that whenever R is ideally closed ($\equiv$ cyclically pure) in an extension algebra S, then R is pure in S. In fact, R has this property if and only if the completion (A, m) of each local ring of R at a maximal ideal has the following two equivalent properties:

(i) For each integer $N > 0$ there is an m-primary irreducible ideal $I_N \subset m^N$.

(ii) Either $\dim A = 0$ and A is Gorenstein or else depth $A \geqslant 1$ and there is no $P \in \operatorname{Ass}(A)$ such that $\dim(A/P) = 1$ and $(A/P) \oplus (A/P)$ is embeddable in A.

It is then shown that if R is a locally excellent Noetherian ring such that either R is reduced (or, more generally, such that R is generically Gorenstein), or such that $\operatorname{Ass}(R)$ contains no primes of coheight $\leqslant 1$ in a maximal ideal, and R is ideally closed in S, then R is pure in S. Matlis duality and the theory of canonical modules are utilized. Module-theoretic analogues of condition (i) above are, of necessity, also analyzed.

Numerous related questions are studied. In the non-Noetherian case, an example is given of a ring extension $R \to S$ such that R is pure in S but $R[[T]]$ is not even cyclically pure in $S[[T]]$.

0. **Introduction.** All rings are commutative, with identity, and modules are unital. Recall that a submodule N of a module M (or the map $N \to M$) over a ring R is called *pure* (respectively, *cyclically pure*) if for every R-module (respectively, every cyclic R-module) E the map $N \otimes E \to M \otimes E$ is injective. A case of particular interest is the one where $N = R$ and $E = S$ is an extension algebra of R: in this case the cyclic purity of R in S asserts that for every ideal I of R, $IS \cap R = I$ (let $E = R/I$ in the definition of cyclic purity). This condition is sometimes expressed by the phrase "R is ideally closed in S". Quite generally, if M/N is finitely presented, then N is pure in M if and only if N is a direct summand of M. We refer the reader to [3], [14], [15], [20, p. 64], and [24] for basic facts about purity.

Our original objective was to give a really useful condition on a Noetherian

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ring R such that whenever R satisfies the condition and is cyclically pure in an R-algebra S, then R is pure in S. As it turns out, the "approximately Gorenstein" condition we discuss below is not only sufficient for this purpose but also necessary, and guarantees the desired conclusion even when S is merely a module, without an algebra structure.

But the definition of "approximately Gorenstein" leaves much to be desired: the hard job is to give a more down-to-earth characterization of this notion. It turns out to be a rather weak (albeit strange) unmixedness condition.

A good deal of our effort will be devoted to proving the existence of m-primary irreducible ideals contained in arbitrarily high powers of the maximal ideals of certain Noetherian local rings (R, m): the existence of such irreducible ideals is a subtle question and is the key to all our difficulties. It turns out to be necessary to consider an analogous question for modules as well (see Theorem 1.12). Matlis duality [17] plays a crucial role at a certain point, and so does the notion of "canonical module". The use of canonical modules has been quite minimized in this write-up: they played a much larger role in the original thinking. It will still be evident (see Example (5.4)) that the existence of canonical modules and the existence of small irreducibles are closely related questions, and we shall later give an example, based on the Ferrand-Raynaud example [5], in which cyclic purity fails to imply purity for a one-dimensional local Noetherian domain R.

We note that some work on the comparison of cyclic purity and purity has been done in [2] and [9] (and cyclic purity is studied in [8]). Where we say that R is "cyclically pure" or "ideally closed" in S, in [9] $R \to S$ is said to satisfy "condition C". Where we refer to the "purity" of $R \to S$, [9] refers to "condition ξ" (essentially, P. M. Cohn's linear equational criterion for purity: see [3] or [20, p. 65]).

1. **The main notions and the main results.** We shall write $\dim R$ for the Krull dimension of R.

Let (R, m) be a Noetherian local ring with $\dim R = n$. We recall ([1], [16]) that R is *Gorenstein* if, equivalently, either:

(i) $\mathrm{id}_R\, R$ is finite (id denotes injective dimension); in this case, $\mathrm{id}_R\, R = n$, or

(ii) some (respectively, every) system of parameters $x_1, \ldots, x_n$ is an R-sequence such that $(x_1, \ldots, x_n)R$ is irreducible, or

(iii) R is Cohen-Macaulay and for some (respectively, every) system of parameters $x_1, \ldots, x_n$ of R, $S = R/(x_1, \ldots, x_n)R$ is an injective S-module.

An arbitrary Noetherian ring R is Gorenstein if all its local rings are.

We are now ready for our most important definition:

(1.1) DEFINITION-PROPOSITION. *A local Noetherian ring (R, m) is approximately Gorenstein if it satisfies either of the following equivalent conditions*:

(i) *For every integer* $N > 0$ *there is an ideal* $I \subset m^N$ *such that* R/I *is Gorenstein.*

(ii) *For every integer* $N > 0$ *there is an m-primary irreducible ideal* $I \subset m^N$.

[For a proof, see (2.1) and (2.2) of §2.]

(1.2) PROPOSITION. *A local Noetherian ring* (R, m) *is approximately Gorenstein if and only if its m-adic completion* $(\hat{R}, \hat{m})$ *is approximately Gorenstein.*

[For a proof, see (2.2) of §2.]

(1.3) DEFINITION. A Noetherian ring R is *approximately Gorenstein* if for every maximal ideal m of R, R_m is approximately Gorenstein.

The "propositions" above are rather trivial. The next proposition is more interesting: it displays the connection between approximately Gorenstein rings and the original problem of getting cyclic purity to imply purity.

(1.4) PROPOSITION. *Let R be a Noetherian ring. The following three conditions on R are equivalent*:

(i) *R is approximately Gorenstein.*

(ii) *R is pure in every extension algebra* $S \supset R$ *such that R is cyclically pure* (*i.e., ideally closed*) *in S.*

(iii) *R is pure in every extension module* $M \supset R$ *such that R is cyclically pure in M.*

[For a proof, see (2.6) of §2.]

Examples of ring extensions $R \subset S$ such that R is Noetherian (even Artinian), R is cyclically pure in S, yet R is *not* pure in S abound. We shall give examples below: see §3 and Example (5.4). The reader may wish to consult [4] and [2] where other examples in which cyclic purity holds but purity fails are given. In the example of [2], R is Artinian and S is a finite R-module.

On the other hand, when R is excellent Noetherian a rather weak unmixedness condition is sufficient for cyclic purity to imply purity if $\dim R \geqslant 2$. By virtue of (1.4) and (1.2) the problem is basically to characterize the approximately Gorenstein rings in the complete local case. If $\dim R = 0$ we have a trivial result:

(1.5) PROPOSITION. *If R is an Artin local ring, then R is approximately Gorenstein if and only if R is Gorenstein.*

[See (2.26).]

If $\dim R \geqslant 1$ we have our first main theorem:

(1.6) THEOREM. *Let* (R, m) *be a complete* (*or even an excellent*) *local ring with*

$\dim R \geqslant 1$. *Then R is approximately Gorenstein if and only if the following two conditions hold*:

(a) $m \notin \mathrm{Ass}(R)$, *i.e.*, $\operatorname{depth} R \geqslant 1$.

(b) *If* $P \in \mathrm{Ass}(R)$ *and* $\dim R/P = 1$, *then* $(R/P) \oplus (R/P)$ *is not embeddable in* R.

[See (5.2) for a proof.]

For basic properties of excellent rings we refer the reader to [18]. We merely mention here that the class of excellent Noetherian rings contains all fields, the integers, all discrete valuation rings of characteristic 0, and is closed under passage to a residue class ring, passage to a ring of quotients with respect to a multiplicative system, and passage to a finitely generated extension algebra. For other properties we shall refer to [18] as needed.

For the purpose of applying (1.6) the following corollary is really our main result (note: by "R is locally excellent" we mean the local rings of R are excellent):

(1.7) THEOREM. *Let R be any locally excellent Noetherian ring and suppose that at least one of the following three conditions holds*:

(1) *R is reduced* (*every nilpotent is* 0), *or*

(2) *R is generically Gorenstein* (*i.e., the total quotient ring of R is a* (*necessarily* 0-*dimensional*) *Gorenstein ring*), *or*

(3) *for every prime* $P \in \mathrm{Ass}(R)$ *and maximal ideal* $m \supset P$, $\dim(R/P)_m \geqslant 2$.

Let S be any extension algebra of R such that R is cyclically pure ($\equiv$ *ideally closed*) *in S. Then R is pure in S. Hence, if S is also a finite R-module, then if R is cyclically pure in S, R is a direct summand of S as an R-module.*

[See (5.3) for a proof.]

This result is not best possible, but conditions (1) or (2) or (3) are the most easily verified in practice. Note that (1) $\Rightarrow$ (2), since if R is reduced its total quotient ring is a finite product of fields. In general, if R is generically Gorenstein its total quotient ring will be a semilocal zero-dimensional Gorenstein ring, and, hence, a finite product of zero-dimensional Gorenstein local rings.

The key point will be to prove that a "good" R has sufficiently many irreducibles. We proceed by working with modules, which is both natural and necessary. Before stating our main result for modules, we give two convenient definitions:

(1.8) DEFINITION. We shall say that a local ring A is an *E-ring* if every domain B which is a homomorphic image of A satisfies the following condition: if $\dim B = 1$, then the generic fiber $(B - \{0\})^{-1}\hat{B}$ of $B \to \hat{B}$ is Gorenstein, while if $\dim B \geqslant 2$ then $\mathrm{Ass}(\hat{B})$ contains no primes of coheight one.

We say that an arbitrary Noetherian ring R is an E-ring if its localization at each maximal ideal is an E-ring.

The key point that we need about excellent rings is simply:

(1.9) PROPOSITION. *A (locally) excellent Noetherian ring is an E-ring. Hence, a complete local ring is an E-ring.*

PROOF. See [18, §34].

(1.10) DEFINITION. Let (R, m) be a local ring and let M be an R-module of finite type. We shall say that M has *small cofinite irreducibles* (S.C.I.) if for every integer $N > 0$ there is an irreducible submodule E of M such that $E \subset m^N M$ and M/E has finite length.

By dim M, where M is an R-module, we mean $\dim(R/\operatorname{Ann}_R M)$. One has, trivially:

(1.11) PROPOSITION. *Let (R, m) be local Noetherian and M an R-module of finite type. Then:*

(a) *M has S.C.I. if and only if $\hat{M}$ has S.C.I. over $\hat{R}$.*

(b) *If $\dim M = 0$, then M has S.C.I. if and only if 0 is irreducible in M.*

(c) *$M = R$ has S.C.I. if and only if R is approximately Gorenstein.*

Our main result on modules having S.C.I. is the following:

(1.12) THEOREM. *Let (R, m) be a local Noetherian ring and let M be an R-module of finite type with $\dim M > 0$. Consider the following condition on M:*

(*) *$m \notin \operatorname{Ass}(M)$ (i.e., $\operatorname{depth} M \geqslant 1$) and if $P \in \operatorname{Ass}(M)$ is such that $\dim(R/P) = 1$, then $(R/P) \oplus (R/P)$ cannot be embedded in M.*

If M has small cofinite irreducibles, then M satisfies (*), while if R is an E-ring then M has S.C.I. if and only if M satisfies (*).

[For a proof, see (4.4), (4.11), and (4.17).]

This result is really the strongest of our theorems: it gives (1.6) and (1.7) at once. Moreover, the following proposition shows that (1.12) is best possible in a certain sense and justifies the notion of an E-ring:

(1.13) PROPOSITION. *Let (R, m) be a local ring and suppose that every R-module M of finite type such that $\dim M > 0$ which satisfies (*) has small cofinite irreducibles. Then R is an E-ring. Thus, R is an E-ring if and only if (*) is equivalent to having S.C.I. for positive dimensional M.*

[For a proof, see (4.12).]

We also note the following easy consequence of (1.6) and (1.7).

(1.14) PROPOSITION. *Let R be an arbitrary Noetherian ring and let x, y be analytic indeterminates over R. Then $R[x,y]$ and $R[[x,y]]$ are approximately Gorenstein.*

Moreover, given a ring extension $R \to S$, where R is Noetherian, $R \to S$ is pure if and only if $R[x,y] \to S[x,y]$ is cyclically pure if and only if $R[[x,y]] \to S[[x,y]]$ is cyclically pure.

[For a proof, see (5.6).]

On the other hand, we note the following result from §6.

(1.15) PROPOSITION. *There exists a pure ring extension $R \to S$ such that $R[[t]] \to S[[t]]$ is not even cyclically pure: in fact principal ideals are not concerned.*

[See (6.1).]

We conclude this summary of our main results with the remark that in §3 it is shown that if R is a 0-dimensional local Noetherian ring *which is not Gorenstein*, it has a sort of "universal" or "generic" finitely generated extension algebra T_R such that $R \to T_R$ is not pure. Moreover, R is cyclically pure but not pure in T_R.

2. Approximately Gorenstein rings, cyclic purity, and purity. The following proposition implies the results in (1.1) and (1.2) of §1.

(2.1) PROPOSITION. *Let (R, m) be a local Noetherian ring, $N > 0$ an integer, and suppose that $I \subset m^N$ and R/I is Gorenstein. Then there exists an ideal J with $I \subset J \subset m^N$ such that J is m-primary and, equivalently:*

(1) *J is irreducible.*

(2) *R/J is Gorenstein.*

(3) $\mathrm{Soc}(R/J) \cong R/m$ (*where* "Soc" *denotes* "*Socle*").

PROOF. Let $k = \dim(R/I)$ and let $x_1, \ldots, x_k \in m$ be such that $\bar{x}_1, \ldots, \bar{x}_k$ is a system of parameters for R/I, where $\bar{\ }$ denotes reduction modulo I. Then $\bar{x}_1^N, \ldots, \bar{x}_k^N$ is also a system of parameters for R/I, and $(R/I)/(\bar{x}_1^N, \ldots, \bar{x}_k^N)$ is a zero-dimensional Gorenstein ring. Hence, we may let $J = I + (x_1^N, \ldots, x_k^N)R$. Q.E.D.

(2.2) COROLLARY. (a) *The conditions given in* (1.1) (*defining "approximately Gorenstein"*) *are equivalent.*

(b) *Gorenstein rings are approximately Gorenstein. A zero-dimensional ring is approximately Gorenstein if and only if it is Gorenstein.*

(c) *A local Noetherian ring (R, m) is approximately Gorenstein if and only if its m-adic completion $(\hat{R}, \hat{m})$ is approximately Gorenstein.*

PROOF. In (1.1) the implication (ii) $\Rightarrow$ (i) is clear, while (i) $\Rightarrow$ (ii) is immediate from (2.1) above. The first statement in (b) is clear from characterization (i) of approximately Gorenstein rings. For the second statement of (b) we need only consider the case of a zero-dimensional Noetherian local ring

(R, m). Choose N so that $m^N = 0$. If R is approximately Gorenstein, we can choose $I \subset m^N$ so that $R/I = R/(0) \cong R$ is Gorenstein.

Finally, statement (c) is clear from characterization (ii) of approximately Gorenstein rings. Q.E.D.

We next want to give a result which contains Proposition (1.4). It will be convenient to make some definitions:

(2.3) DEFINITION. If R is a Noetherian ring, $\mathfrak{I}(R) = \{I \subset R: I$ is an irreducible ideal and $\mathrm{Rad}(I)$ is a maximal ideal of $R\}$.

(2.4) DEFINITION. If R is a Noetherian ring, $\mathbb{S}(R)$ (or simply $\mathbb{S}$) denotes the R-algebra $\prod_{I \in \mathfrak{I}(R)} R/I$.

We also note the following trivial fact:

(2.5) LEMMA. *Let R be a Noetherian ring. If $r \in R$, $r \neq 0$, and I is maximal with respect to not containing r, then $I \in \mathfrak{I}(R)$. Hence, for any Noetherian ring R, $\bigcap_{I \in \mathfrak{I}(R)} I = (0)$, i.e., $R \to \mathbb{S}(R)$ is injective.*

The proof is left to the reader. We are now ready for the main result of this section:

(2.6) THEOREM. *Let R be Noetherian ring. Then $R \to \mathbb{S}(R)$ is ideally closed, and the following four conditions are equivalent:*

(i) *R is approximately Gorenstein.*

(ii) *$R \to \mathbb{S}(R)$ is pure.*

(iii) *For every extension algebra S of R, if R is ideally closed ($\equiv$ cyclically pure) in S, then R is pure in S.*

(iv) *For every extension module M of R, if $R \to M$ is cyclically pure, then R is pure in M.*

PROOF. We shall show that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv). (iv) $\Rightarrow$ (iii) is trivial, while (iii) $\Rightarrow$ (ii) is immediate if we know that $R \to \mathbb{S}$ is ideally closed. To see this suppose, to the contrary, that $I \subset R$, $r \in R - I$, and $r \in I\mathbb{S}$. Choose $J \supset I$ maximal with respect to not containing r. By (2.5), $J \in \mathfrak{I}(R)$. If we compose $R \to \mathbb{S} \to^p R/J$, where p is the product projection for the J-coordinate, then since $r \in I\mathbb{S} \subset J\mathbb{S}$, the image of r is 0 in R/J, contradicting $r \notin J$. Thus, $R \to \mathbb{S}(R)$ is ideally closed, and (iii) $\Rightarrow$ (ii).

To see that (ii) $\Rightarrow$ (i), assume that $R \to \mathbb{S}(R)$ is pure and let m be a given maximal ideal of R. Let N be a given integer > 0. Let $K = R/m$ and let E denote the injective hull of K as a module over $B = R/m^n$. Then, using the basic facts about Matlis duality [17], we have that E has finite length as a B-module and, hence, E is finitely generated ($\Rightarrow$ finitely presented) as an R-module. Moreover, $\mathrm{Hom}_R(E, E) = \mathrm{Hom}_B(E, E) \cong B$ (the last isomorphism is the crucial consequence of Matlis duality which we need here), where under $B \cong \mathrm{Hom}_B(E, E)$ the element b maps to multiplication by b.

Since $R \to \mathbb{S}$ is pure, the map $E \to E \otimes_R \mathbb{S} = E \otimes_R \prod_I (R/I)$ is *injective.* Now, quite generally, if R is any commutative ring, E is a *finitely presented* R-module, and $\{Q_\lambda\}_{\lambda \in \Lambda}$ is an arbitrary family of R-modules, the natural map

$$E \otimes_R \left(\prod_\lambda Q_\lambda\right) \to \prod_\lambda (E \otimes_R Q_\lambda)$$

is an isomorphism. [The map takes $e \otimes f$, where $f\colon \Lambda \to \bigcup_\lambda Q_\lambda$ is such that $f(\lambda) \in Q_\lambda$, to g, where $g(\lambda) = e \otimes f(\lambda)$.] This gives a natural transformation from the right exact functor $F = \cdot \otimes_R (\prod_\lambda Q_\lambda)$ to the right exact functor $G = \prod_\lambda (\cdot \otimes_R Q_\lambda)$. Hence, to check that $F(E) \to G(E)$ is an isomorphism for finitely presented E, it suffices to check the case $E = R^m$. But then the natural transformation is given by $F(E) = R^m \otimes_R (\prod_\lambda Q_\lambda) \cong (\prod_\lambda Q_\lambda)^m \cong \prod_\lambda (Q_\lambda^m) = \prod_\lambda (R^m \otimes_R Q_\lambda) = G(E)$. Hence, $E \otimes_R \mathbb{S} \cong \prod_I (E \otimes_R R/I) \cong \prod_I (E/IE)$, where $I \in \mathcal{I}(R)$, and we have that $E \to \prod_I (E/IE)$ is injective. Let u generate the copy of K in E, i.e., $Ru \cong K \hookrightarrow E$. Then we can choose $I \in \mathcal{I}(R)$ such that $K \cong Ru \not\subset IE$. But Ru is contained in every nonzero submodule of E. Hence, we can choose $I \in \mathcal{I}(R)$ such that $IE = 0$. But then $I \subset m^N$ as required, for if $r \in I - m^N$ then $\bar{r} \neq 0$ in $B = R/m^N$ ($^-$ denotes reduction module m^N) and, hence, by Matlis duality, multiplication by r or $\bar{r}$ gives a *nonzero* endomorphism of E $(B \cong \operatorname{Hom}_B(E, E))$. This says that $rE \neq 0$ and so $IE \neq 0$, a contradiction. This completes the proof that (ii) $\Rightarrow$ (i).

It remains to show that (i) $\Rightarrow$ (iv). Suppose that R is approximately Gorenstein and that $0 \to R \to M$ is cyclically pure. We must show that for every R-module E, $E = R \otimes E \to M \otimes E$ is injective. Since E is a direct limit of finitely generated modules, we may assume that E is finitely generated. Suppose $e \neq 0$ in E but $1 \otimes e$ is 0 in $M \otimes E$. Let m be a maximal ideal of R which contains $\operatorname{Ann}_R Re$. $0 \to Re \to E$ is injective, and hence $0 \to (Re)_m \to E_m$ is injective, and $(Re)_m \neq 0$. Choose N such that $e/1$ is not in $m^N E_m$: this is possible since $e/1 \neq 0$ and $\bigcap_t m^t E_m = 0$. Then the image of e in $E_m/m^N E_m \cong E/m^N E$ is not 0. We have a commutative diagram:

$$\begin{array}{ccc} E & \longrightarrow & E \otimes M \\ \downarrow & & \downarrow \\ E/m^N E & \longrightarrow & (E/m^N E) \otimes M \end{array}$$

Since e maps to 0 in $E \otimes M$, its image in $E/m^N E$ (is nonzero) but maps to 0 in $(E/m^N E) \otimes M$. Thus, purity fails for a module $\overline{E}$ which is killed by m^N for a certain m, N. But this is impossible: choose $I \subset m^N$ so that I is m-primary and irreducible (R is approximately Gorenstein, which permits us to do this).

Then $m^N \bar{E} = 0 \Rightarrow I\bar{E} = 0 \Rightarrow \bar{E} \cong R/I \otimes_R \bar{E}$; moreover, $B = R/I$ is zero-dimensional Gorenstein and, hence, B-injective. Since $0 \to R \to M$ is cyclically pure, $0 \to B \to B \otimes M$ is exact, and then since B is B-injective, B is a direct summand of $B \otimes M$ as B-modules, whence $0 \to E \otimes_B B \to E \otimes_B (B \otimes_R M)$ is injective, and since $IE = 0$, this sequence may be identified with $0 \to E \to E \otimes_R M$. Q.E.D. for (i) $\Rightarrow$ (iv) and Theorem 2.6.

(2.7) REMARK. The fact that R is approximately Gorenstein does not imply that R_P is approximately Gorenstein for every P. It is easy to see this if we allow ourselves the use of the characterization of Theorem 1.6. Let $R = K[[x_1, x_2, y_1, y_2]]/(y_1, y_2)^2$, where K is any field. Then R is approximately Gorenstein, because the unique prime P in $\mathrm{Ass}(R)$ (which is generated by the images of y_1, y_2) is of coheight 2. But R_P is zero-dimensional and not Gorenstein, and hence not approximately Gorenstein.

(2.8) REMARK. Suppose we had allowed irreducibles whose primes are nonmaximal in the construction of $\mathbb{S}$ and had then tried to prove, instead of (ii) $\Rightarrow$ (i) of (2.6), that given (ii), R_P is approximately Gorenstein for every prime P of R. Then we immediately run into the stumbling block that the injective hull of R_P/PR_P as an (R_P/P^nR_P)-module is not of finite type over R, and we cannot distribute $\otimes$ over the infinite product of modules as in the proof of (2.6).

(2.9) REMARK. The implication (i) $\Rightarrow$ (iv) is essentially given, at least implicitly, in the local case, in [12]. The ideas of [12] motivated our work here, while [12] in turn was motivated by the relationship between results on contractedness and the existence of big Cohen-Macaulay modules: this relationship is discussed in [13].

3. Generic nonpure extension algebras of Artin local rings.

(3.1) PROPOSITION. *Let (R, m) be an Artin local ring, let $K = R/m$, and let A be an s by t matrix with entries in m such that* $\mathrm{Coker}(A\colon R^s \to R^t)$ *is an injective hull E for K (we may obtain such a matrix from a minimal free R-resolution of an injective hull of K). Let $X = [X_1 \cdots X_s]$ be an $s \times 1$ matrix of indeterminates, let $S = R[X_1, \ldots, X_s]$, let $p \in R^t$ map to an element of E which generates the copy of K in E, and let J be the ideal of S generated by the entries of $p - XA$. Let $T_R = S/J$. Then:*

(a) *T_R is a finitely generated R-algebra. Its R-algebra structure is independent of the choice of A or P.*

(b) *$R \to T_R$ is not pure, and is "generic" with respect to this property in the sense that $h\colon R \to U$ is not pure if and only if there is a homomorphism $T_R \to U$ such that $h = (T_R \to U) \circ (R \to T_R)$.*

(c) *If R is not Gorenstein, $R \to T_R$ is injective and cyclically pure (R is ideally* *closed in* T_R). *If R is Gorenstein*, $s = 0$, $S = R$, *and* $T_R \cong R/\operatorname{Ann}_R m$; $R \to T_R$ *is not injective.*

PROOF. (a) Consider one choice of A, p. Any other choice of A has the form BAC, where B is s by s invertible, C is t by t invertible, and then we may *initially* choose the new p to be pC. Let $X' = XB$. Then the entries $X'_1, \ldots, X'_s$ of X' are indeterminates over R and $S = R[X'_1, \ldots, X'_s]$. The new algebra T'_R obtained is S/J', where J' is generated by the entries of $pC - X(BAC) = pC - (XB)(AC) = (p - (XB)A)C$, and since C is invertible J' is generated by the entries of $p - X'A$ and

$$T'_R = R[X'_1, \ldots, X'_s]/(\text{the entries of } p - X'A) \cong T_R.$$

Now suppose that A is fixed but we vary p. Any choice of p has the form $p' = ap + DA$ where a is an invertible element of R and D is a 1 by s matrix of elements of R, say $D = [D_1 \cdots D_s]$. In this case $T'_R = S/J'$, where J' is generated by the entries of $ap + DA - XA = a(p - a^{-1}(X - D)A)$, and hence J' is generated by the entries of $p - X'A$, where, now, $X' = a^{-1}(X - D)$. Since the $X'_i = a^{-1}(X_i - D_i)$ are algebraically independent over R and generate S, we again have $T'_R \cong T_R$.

Of course, it is obvious that T_R is finitely generated over R.

(b) Let $\phi\colon K \to E$ be the map which takes $1 + m$ to the image $\bar{p}$ of p. By [14, Proposition 6.11], $h\colon R \to U$ is pure if and only if the induced map $K \otimes U \to E \otimes U$ is injective, i.e., if and only if $\bar{p} \otimes 1$ is nonzero in $E \otimes U$. Since $E = \operatorname{Coker} A$ (over R), $E \otimes U \cong \operatorname{Coker} h(A)$ (over U), and $\bar{p} \otimes 1$ is represented by the image of $h(p)$ in $\operatorname{Coker} h(A)$. Then, h is not pure $\Leftrightarrow$ $h(p)$ represents 0 in $\operatorname{Coker} h(A) \Leftrightarrow h(p)$ is in the row space of $h(A) \Leftrightarrow$ there exists an $s \times 1$ matrix $\bar{X}$ over U such that $h(p) - \bar{X}h(A) = 0 \Leftrightarrow$ there is a homomorphism $S = R[X_1, \ldots, X_s] \to U$ extending $h\colon R \to U$ (corresponding to $X \mapsto \bar{X}$) such that $p - XA$ maps to $0 \Leftrightarrow h\colon R \to U$ factors through $S/J = T_R$, as required.

(c) If R is not Gorenstein, then by (2.2b) R is not approximately Gorenstein, and by Theorem 2.6, $R \to \mathbb{S}(R)$ is injective, ideally closed, but not pure. Hence, this map has a factorization $R \to T_R \to \mathbb{S}(R)$, by part (b) here. This implies $R \to T_R$ is injective and ideally closed.

If R is Gorenstein, $E = R$ and the minimal resolution is $0 \to R \to R \to 0$, i.e., $s = 0, t = 1$. We interpret this to mean that $S = R$ (there are 0 indeterminates). p is then, simply, an element which generates the unique copy of $K \cong \operatorname{Ann}_R m$ in R, and we regard the product of the 1 by 0 matrix X and the 0 by 1 matrix A to be the 1×1 matrix $[0]$, so that J is generated by $p - 0 = p$. Q.E.D.

(3.2) REMARK. If (R, m) is local one cannot get a single algebra to play the role of T_R, because the injective hull E of R/m is countably generated rather than finitely generated.

(3.3) EXAMPLE. Let $R = K[X, Y]/(X, Y)^2 = K[x,y]$ $(= K + Kx + Ky)$, where K is any field. In this case, we may take

$$A = \begin{bmatrix} y & 0 \\ 0 & x \\ x & -y \end{bmatrix}$$

and $p = [x\ 0]$ $(\equiv [0\ y])$. Then

$$p - XA = [x - X_1 y - X_3 x, -X_2 x + X_3 y],$$

and

$$T_R = R[X_1, X_2, X_3]/(x - X_1 y - X_3 x, -X_2 x + X_3 y).$$

Then R is not pure in T_R, and since R is not Gorenstein R is ideally closed in T_R, by Proposition 3.1. But it is easy to see this directly: except for (0) and m (which are easily checked separately), every proper ideal of R is irreducible and is, in fact, the kernel of a surjective homomorphism $h\colon R \to K[t] \cong K[T]/(T^2)$. h is uniquely determined by elements $a, b \in K$, where $(a,b) \neq (0,0)$, and $h(x) = at$, $h(y) = bt$. The assertion that $((\operatorname{Ker} h)T_R) \cap R = \operatorname{Ker} h$ is equivalent to the injectivity of the map

$$K[t] \to K[t][X_1, X_2, X_3]/(at - X_1 bt - X_3 at, -X_2 at + X_3 bt)$$

induced by substituting $x = at$, $y = bt$. If $a = 0$ this injectivity is clear: we may further compose with a map that kills $\overline{X}_1, \overline{X}_2, \overline{X}_3$. If $a \neq 0$ we may compose with a map which sends $\overline{X}_1$ to 0, $\overline{X}_3$ to 1, and $\overline{X}_2$ to ba^{-1}. In either case the target ring may be retracted to $K[t]$.

4. The existence of small cofinite irreducibles. In this section, which is the heart of the paper, we prove the key theorem, (1.12). We proceed by giving a sequence of lemmas and propositions which eventually yields the desired result.

Throughout the rest of this section, unless otherwise specified, (R, m) denotes a Noetherian local ring and $K = R/m$. $\hat{}$ denotes m-adic completion. M and E always denote R-modules of finite type. (Some of these hypotheses may be reiterated, for emphasis, in the statements of theorems.)

(4.1) PROPOSITION. (a) *If* $\dim M = 0$, *M has small cofinite irreducibles if and only if* 0 *is irreducible in M, i.e., $M = 0$ or $M \neq 0$ and* $\operatorname{Hom}_R(K, M) \cong K$.

(b) *M has S.C.I. over R if and only if $\hat{M}$ has S.C.I. over $\hat{R}$.*

(c) *If $I \subset \operatorname{Ann}_R M$, then M has S.C.I. over R/I if and only if M has S.C.I. over R.*

The proof is utterly straightforward and is left to the reader.

(4.2) PROPOSITION. *M has small cofinite irreducibles if and only if for every integer* $N > 0$ *there is a submodule* $M_N \subset m^N M$ *such that* M/M_N *has S.C.I.*

PROOF. "Only if" is clear, from the definition of having S.C.I. and (4.1a). To prove "if", let $N > 0$ be given and choose $M_N \subset m^N M$ such that M/M_N has S.C.I. Choose $E \subset m^N(M/M_N)$ such that $(M/M_N)/E$ has finite length and E is irreducible in M/M_N (i.e., 0 is irreducible in $(M/M_N)/E$). Let M'_N be the inverse image of E in M, so that $M/M'_N \cong (M/M_N)/E$. Then $M'_N \subset m^N M + M_N \subset m^N M$ and M'_N is cofinite (i.e., M/M_N has finite length), while M'_N is irreducible in M because 0 is irreducible in $M/M'_N \cong (M/M_N)/E$. Q.E.D.

(4.3) PROPOSITION. *Suppose that M has S.C.I. Then every submodule E of M has S.C.I.*

PROOF. By the Artin-Rees lemma there is a positive integer c such that for every integer $t > 0$, $(m^{t+c}M) \cap E \subset m^t E$. Let $N > 0$ be given and choose $M' \subset m^{N+c}M$ such that M/M' has finite length and M' is irreducible in M. Let $E_N = M' \cap E$. Then $E_N \subset (m^{N+c}M) \cap E \subset m^N E$. Moreover, we have an injection $E/E_N \hookrightarrow M/M'$. Since M/M' has finite length and 0 is irreducible in it, every submodule of M/M' has the same property. Thus, E_N is cofinite and irreducible in E. Q.E.D.

We are now ready to begin the proof of the necessity of condition (*) for a module to have S.C.I. (see (1.12)). First:

(4.4) PROPOSITION. *If M has S.C.I. and* $\dim M \geqslant 1$ *then* $\operatorname{depth} M \geqslant 1$, *i.e.*, $m \notin \operatorname{Ass}(M)$.

PROOF. If $\operatorname{depth} M = 0$, choose $x \neq 0$ in M such that $mx = 0$. Since $x \neq 0$ we can choose $N > 0$ such that $x \notin m^N M$. Choose $M' \subset m^{N+1}M$ such that M' is cofinite and irreducible in M. Since $x \notin M'$ and $mx = 0$, $x + M'$ generates $\operatorname{Soc}(M/M') = \operatorname{Ann}_{M/M'} m$ ($\cong K$, since M' is irreducible). Since $\dim M \geqslant 1$, $m^N M \neq m^{N+1}M$. Let $y \in m^N M - m^{N+1}M$. Since $y \notin M'$, $y + M'$ has a multiple equal to $x + M'$, i.e., there exists $r \in R$ such that $x - ry \in M' \subset m^{N+1}M$. Since $y \in m^N M$, $x \in m^N M + m^{N+1}M = m^N M$, a contradiction. Q.E.D.

Before continuing the proof of the necessity of (*), we recall some needed facts about canonical modules in the local case.

(4.5) DISCUSSION. Let (R, m) be a Cohen-Macaulay local ring, with $\dim R = n$. A *canonical module E* for R is an R-module of finite type satisfying the following condition: if $x_1, \ldots, x_n$ is a system of parameters for R (equivalently, a maximal R-sequence) then $x_1, \ldots, x_n$ is an E-sequence and $E/(x_1, \ldots, x_n)E$ is an injective hull for K regarded as an $(R/(x_1, \ldots, x_n)R)$-module. E is unique up to nonunique isomorphism, if it exists. However, the natural map $R \to \operatorname{Hom}_R(E, E)$ is an isomorphism, so that the isomorphism between two canonical modules is unique up to multiplication by a unit of R.

A Cohen-Macaulay ring R possesses a canonical module if and only if R is a homomorphic image of a Gorenstein local ring [19], [7], [6]: a complete local Cohen-Macaulay ring always has one. R is Gorenstein if and only if it is a canonical module for itself. See [22] and [23] for more information. (Note: A "canonical module" is a "rank one Gorenstein module" in the terminology of some authors.)

We note that if R is of finite type over (perhaps a homomorphic image of) a Gorenstein local ring S of dimension $q \geqslant n$, then $\operatorname{Ext}_S^{q-n}(R,S)$ is a canonical module for R.

If E is a canonical module for R and $x \in R$ is not a zerodivisor, then x is not a zerodivisor on E and E/xE is a canonical module for R/xR.

If P is a prime ideal of R, then E_P is a canonical module for R_P.

We refer the reader to [7], [10], [11], [22], and [23] for further information and details.

We also need the following fact from [11]:

(4.6) FACT. *Let (R,m) be a Cohen-Macaulay ring which possesses a canonical module E. Let $x_1, \ldots, x_n$ be a system of parameters for R. Then*

$$W = \operatorname{inj}\lim_t(\cdots \to E/(x_1^t,\ldots,x_n^t)E \xrightarrow{x_1\cdots x_n} E/(x_1^{t+1},\ldots,x_n^{t+1})E \to \cdots)$$

is an injective hull for K over R.

PROOF. $E/(x_1^t,\ldots,x_n^t)E$ is an injective hull of K over $R/(x_1^t,\ldots,x_n^t)R$, and, since $x_1, \ldots, x_n$ is an E-sequence, the maps are injective. It follows that the modules in the sequence fit together to give the injective hull of K over R. (Alternatively, by [11, Theorem 2.3], $W = H_m^n(E) = H_m^n(\hat{E})$, is an injective hull of K over R or $\hat{R}$ by local duality.) Q.E.D.

(4.7) REMARK. If R is an Artin local ring, a canonical module for R is precisely the same as an injective hull for K as an R-module.

Recall that a (not necessarily local) Noetherian ring R is *generically Gorenstein* if its total quotient ring is Gorenstein. We need the following useful though elementary:

(4.8a) FACT. *Let R be a generically Gorenstein Cohen-Macaulay local ring which possesses a canonical module E. Then R and E can each be embedded in the other. Moreover, E is isomorphic as an R-module with an ideal of R which may be taken to be R precisely if R is Gorenstein and which otherwise has pure height one.*

PROOF. Let T denote the multiplicative system of nonzerodivisors in R; $T = R - \bigcup_{i=1}^m p_i$, where $p_1, \ldots, p_m$ are the minimal primes of R. $T^{-1}R$ is a zero-dimensional Gorenstein ring and $T^{-1}R \cong \prod_{i=1}^m R_{p_i}$. $T^{-1}E \cong \prod_{i=1}^m E_{p_i}$, and since each R_{p_i} is Gorenstein and E_{p_i} is a canonical module for R_{p_i}, we have $E_{p_i} \cong R_{p_i}$. Hence, $T^{-1}R \cong T^{-1}E$. Now, we have injections $R \hookrightarrow T^{-1}R$, $E \hookrightarrow T^{-1}E$. Hence, the isomorphism $T^{-1}R \cong T^{-1}E$ provides injections $R \hookrightarrow T^{-1}E$, $E \hookrightarrow T^{-1}R$. Since R and E are R-modules of finite type, we may choose nonzerodivisors $a, b \in R$ such that the image of aR ($\cong R$) in $T^{-1}E$ is contained in E, and the image of bE ($\cong E$) in $T^{-1}R$ is contained in R. Thus, we have injections $R \hookrightarrow E$ and $E \hookrightarrow R$. Let I be an ideal of R such that $I \cong E$. Let P be an associated prime of R and let $A = R_P$. Then IA is a canonical module for A, and we cannot have height $P \geqslant 2$, for then $\operatorname{depth}_A A \geqslant 2$, $\operatorname{depth}_A IA \geqslant 2$, and so $\operatorname{depth}_A A/IA \geqslant 1$, while $P \in \operatorname{Ass}(R/I) \Rightarrow \operatorname{depth}_A A/IA = 0$. Q.E.D.

(4.8b) REMARKS. *If R is Cohen-Macaulay and has a canonical module, then if R is generically Gorenstein, R is approximately Gorenstein.* For the canonical module E always has S.C.I.: if $x_1, \ldots, x_n$ is a system of parameters for R, then $(x_1^N, \ldots, x_n^N)E$ is irreducible in E. If R is generically Gorenstein, we can embed $R \hookrightarrow E$, and then R has S.C.I., i.e., R is approximately Gorenstein. See Example (5.4).

We are almost ready to prove the necessity of condition (*) for a module to have S.C.I. First, we want to make some observations.

(4.9) DISCUSSION. Let R be Noetherian (not necessarily local). If P is a prime of R, we write $\mu_i(P, M)$ for

$$\dim_{\kappa(P)} \operatorname{Ext}^i_{R_P}(\kappa(P), M_P),$$

where $\kappa(P) = R_P/PR_P$: $\mu_i(P, M)$ is the ith Bass number (and can also be characterized as the number of copies of the injective hull over R (or over R_P) of $\kappa(P)$ occurring in the ith term in a minimal injective resolution of M [1]. Here, we shall be concerned solely with the numbers $\mu_0(P, M)$, or, briefly, $\mu(P, M)$. We note the following trivial facts (the proofs are left to the reader):

(4.10) LEMMA. *Let R be Noetherian (not necessarily local).*

(a) $\mu(P, M) = \mu(PR_P, M_P)$.

(b) $\mu(P, M) > 0 \Leftrightarrow P \in \operatorname{Ass}(M)$.

(c) *$\mu(P, M)$ is the torsion-free rank of any maximal (R/P)-torsion-free submodule of M.*

(d) *$\mu(P, M)$ is the rank of any maximal (R/P)-free submodule of M.*

(e) *If P is minimal in R, R_P is Gorenstein if and only if $\mu(P, R) = 1$.*

(4.11) THEOREM. *If M has S.C.I. and P is a prime of R such that* $\dim(R/P) = 1$, *then $\mu(P, M) \leqslant 1$. In other words, $(R/P) \oplus (R/P)$ cannot be embedded in M.*

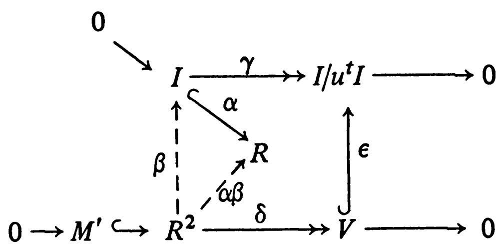
PROOF. If not, we can choose a prime Q (of $\hat{R}$) $\in \operatorname{Ass}(\hat{R}/P)$ such that $\dim \hat{R}/Q = 1$. Then we have an embedding $\hat{R}/Q \hookrightarrow \hat{R}/\hat{P}$ and hence an embedding

$$(\hat{R}/Q) \oplus (\hat{R}/Q) \hookrightarrow (\hat{R}/\hat{P}) \oplus (\hat{R}/\hat{P}) \cong ((R/P) \oplus (R/P))\hat{\ } \hookrightarrow \hat{M}.$$

Thus, there is no loss of generality in assuming that R is complete. By (4.3), $(R/P) \oplus (R/P)$ will itself have S.C.I.: by (4.1c), we may replace R by R/P.

Thus, it suffices to show that if R is a complete one-dimensional local domain, then $R \oplus R$ does not have S.C.I.

Let $u \in m - \{0\}$. Since $\dim R = 1$, we can choose N such that $m^N \subset uR$. We shall show that if M' is cofinite and irreducible in R^2, then $M' \not\subset m^N(R^2)$. To see this, let $V = R^2/M'$. Since V is an essential extension of K (of finite length), we have an injection $V \hookrightarrow W$, where W is the injective hull of K. By (4.6), if I denotes an ideal R which is a canonical module for R (cf. (4.8)), then $W = \text{proj}\lim_t I/u^tI$, and since V has finite length it follows that for some t we have an injection $V \to I/u^tI$. We then have the following commutative diagram in which the sequences are exact:



where α is simply the inclusion of I in R. We can choose β (and then, uniquely, $\alpha\beta$) so that the diagram commutes, since γ is surjective and R^2 is free. Let a, b be the images of the free generators for R^2 in R (under $\alpha\beta$). We may choose a largest possible nonnegative integer h such that $a = u^h a'$, $b = u^h b'$ with $a', b' \in R$. Since h is as large as possible, either $a' \notin uR$ or $b' \notin uR$. Hence $(b', -a') \notin m^N R^2$ (for $m^N \subset uR$). But $\alpha\beta((b', -a')) = b'a - a'b = 0$. Since α is injective, $\beta((b', -a')) = 0$ and so $\gamma\beta((b', -a')) = 0$. But $\gamma\beta = \varepsilon\delta$, so that $\varepsilon\delta((b', -a')) = 0$, and so, since ε is injective, $\delta((b', -a')) = 0$, i.e., $(b', -a') \in M'$. Thus, $M' \not\subset m^N R^2$. Q.E.D.

(4.4) and (4.11) together yield the necessity of the condition (*) for an M with $\dim M > 0$ to have S.C.I.: part of the statement of Theorem (1.12).

We now have enough information to prove the first statement in Proposition (1.13).

(4.12) PROPOSITION. *Suppose that every M with $\dim M > 0$ which satisfies (*) has S.C.I. Then R is an E-ring.*

PROOF. Suppose, to the contrary, that R is not an E-ring. There are two possibilities. One is that for some prime P of R, $\dim R/P \geqslant 2$ and $\text{Ass}(R/P)\hat{\ }$ contains primes of coheight 1. Let $Q \in \text{Ass}(R/P)\hat{\ }$ be such that $\dim \hat{R}/Q = 1$.

Then $(R/P) \oplus (R/P)$ satisfies $(*)$, but its completion has $\hat{R}/Q \oplus \hat{R}/Q$ as a submodule and does not. Hence, by (4.11), $((R/P) \oplus (R/P))\hat{}$ does not have S.C.I., and so neither does $(R/P) \oplus (R/P)$.

The second possibility is that for some prime P of R with $\dim(R/P) = 1$, the ring $(R - P)^{-1}(R/P)\hat{}$ is *not* Gorenstein. This means that for some minimal prime Q of $\hat{P}$, $((R/P)\hat{}\,)_Q$ is not Gorenstein. Since this ring is zero-dimensional, this says (cf. (4.10e)) that $\mu(Q/\hat{P}, (R/P)\hat{}\,) \geqslant 2$, i.e., $(\hat{R}/Q) \oplus (\hat{R}/Q)$ is embeddable in $(R/P)\hat{}$. Thus, by (4.11), $(R/P)\hat{}$ does not have S.C.I. But R/P satisfies $(*)$. Thus, in either case, we have a contradiction. Q.E.D.

Our remaining objective is the result that if R is an E-ring and $\dim M \geqslant 1$, then $(*)$ is sufficient for M to have S.C.I. We need a refined version of primary decomposition for modules [4, Chapter I, §B].

First, recall that if P is a prime of R (not necessarily local) then M is "P-coprimary" if, equivalently, (i) $\text{Ass}(M) = \{P\}$ or (ii) $\text{Rad Ann}\, M = P$, and for every $r \in R$ either r is not a zerodivisor on M or $r \in P$ (in which case r is nilpotent on M).

(4.13) Lemma. *Let R be a (not necessarily local) Noetherian ring and M an R-module of finite type. Let* $\text{Ass}(M) = \{P_1, \ldots, P_h\}$ *and let* $\nu_i = \mu(P_i, M)$, $1 \leqslant i \leqslant h$. *For each i, let M_i be a copy of $(R/P_i)^{\nu_i}$ in M.*

(a) *Let $M^* = \sum_{i=1}^h M_i$. Then the sum is direct. Moreover, M is an essential extension of M^*.*

Since $M_i \cap \sum_{j \neq i} M_j = 0$, for each i we may extend $\sum_{j\neq i} M_j$ to a submodule E_i of M maximal with respect to being disjoint from M_i. Let $M' = \sum \oplus_{i=1}^h (M/E_i) = \prod_{i=1}^h (M/E_i)$. The maps $M \to M/E_i$ induce a map $M \to M'$.

(b) *For each i, M/E_i is an essential extension of the image of M_i, and, hence,* $\text{Ass}(M/E_i) = \{P_i\}$, *i.e., M/E_i is P_i-coprimary and E_i is P_i-primary. Moreover,* $\mu(P_i, M/E_i) = \nu_i$.

(c) *The map $M \to M'$ is injective.*

(d) $\bigcap_{i=1}^h E_i = 0$; *in fact, this is an irredundant primary decomposition of* 0 *in* M.

Proof. (a) If V_1, V_2 are submodules of M such that $\text{Ass}(V_1) \cap \text{Ass}(V_2) = \varnothing$, then $V_1 \cap V_2 = 0$ and $V_1 + V_2 = V_1 \oplus V_2$, for $\text{Ass}(V_1 \cap V_2) \subset \text{Ass}(V_1) \cap \text{Ass}(V_2)$. By induction, if $V_1, \ldots, V_t$ are submodules of M and the sets $\text{Ass}(V_i)$ are pairwise disjoint, $V_1 + \cdots + V_t = V_1 \oplus \cdots \oplus V_t$. Since $\text{Ass}(M_i) = \{P_i\}$, $M^* = \sum_{i=1}^h M_i = \sum \oplus_{i=1}^h M_i$. To see that the extension $M^* \to M$ is essential, let $w \in M - \{0\}$ be given: we must show that $Rw \cap M^* \neq 0$. First, we replace w by a multiple whose annihilator is prime. Thus, we may assume $\text{Ann}_R\, w = P_i$ for a certain i, and $Rw \cong R/P_i$. But then, $Rw \cap M^* = 0$

$\Rightarrow (Rw) \cap M_i = 0$ and then $Rw + M_i \cong Rw \oplus M_i \cong (R/P_i)^{\nu_i+1}$, contradicting the definition of ν_i. This proves (a).

(b) Since $E_i \cap M_i = 0$, we have an injection $M_i \to M/E_i$: the maximality of E_i says precisely that this is an essential extension. The rest is straightforward.

(c) Since M is an essential extension of its submodule $M^* = \sum \oplus_i M_i$, it suffices to show that the restriction $M^* \to M'$ is injective, i.e., that $\phi: \sum \oplus_i M_i \to \prod_i M/E_i$ is injective. If $j \neq i$, $M_j \subset E_i$ by construction of E_i, so that each summand M_i maps to 0 in all coordinates except the ith. Thus, writing $\prod_i (M/E_i) = \sum \oplus_i (M/E_i)$ we see that ϕ is the direct sum of the maps $M_i \to M/E_i$, which we already know are injective.

(d) The injectivity of $M \to \prod_i (M/E_i)$ says precisely that $\bigcap_i E_i = 0$, while from (b), E_i is P_i-primary. Q.E.D.

(4.14) Lemma. *Let R be a (not necessarily local) Noetherian ring, P a prime ideal of R, and M a P-coprimary module with $\mu(P, M) = \nu$. Then there exist ν P-coprimary modules $M_1, \ldots, M_\nu$ such that $\mu(P, M_i) = 1$, $1 \leqslant i \leqslant \nu$, and M can be embedded in $\sum \oplus_{i=1}^{\nu} M_i$.*

Proof. We may replace R by $R/\operatorname{Ann} M$ without loss of generality, and so assume that P is the unique minimal prime of R. Now, suppose we can solve the problem over R_P, and embed $M_P \hookrightarrow \sum \oplus_{i=1}^{\nu} M_i^*$, where the M_i^* are suitable R_P-modules. Each M_i^* has the form $(M_i)_P$ for a suitable R-module of finite type M_i, and M_i may be chosen P-coprimary as well. Since M is of finite type, for a suitable $s \in R - P$ the image of $sM \cong M$ in $\sum \oplus_{i=1}^{\nu} M_i^* = (\sum \oplus_{i=1}^{\nu} M_i)_P$ will be contained in $\sum \oplus_{i=1}^{\nu} M_i$, and we are done.

Thus, we may assume that (R, P) is a zero-dimensional local ring, say with residue class field $K = R/P$, and, as in (4.13a), M is an essential extension of $K^\nu \subset M$. Now, let $M_1^*, \ldots, M_\nu^*$ each be the injective hull of K as an R-module. The maps $K \hookrightarrow M_i^*$ give rise to an injection $K^\nu \to \sum \oplus_{i=1}^{\nu} M_i^*$ which, since $\sum \oplus_{i=1}^{\nu} M_i^*$ is R-injective, extends to a map $M \to \sum \oplus_{i=1}^{\nu} M_i^*$; moreover, since $K^\nu \subset M$ is essential, this map is injective. Q.E.D.

(4.15) Proposition. *Let R be a (not necessarily local) Noetherian ring and let M be an R-module of finite type with* $\operatorname{Ass}(M) = \{P_1, \ldots, P_h\}$ *and $\mu(P_i, M) = \nu_i$. Then M can be embedded in a direct sum of $\sum_{i=1}^{h} \nu_i$ R-modules $\sum \oplus_{1 \leqslant i \leqslant h; 1 \leqslant j \leqslant \nu} M_{ij}$, where M_{ij} is a P_i-coprimary module with $\mu(P_i, M_{ij}) = 1$, $1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant \nu_i$.*

Proof. By (4.13) we can embed $M \to \sum \oplus_{i=1}^{h} M/E_i$ where M/E_i is P_i-coprimary and $\mu(P_i, M/E_i) = \nu_i$, and by (4.14) we can embed M/E_i in a direct sum $\sum \oplus_{j=1}^{\nu_i} M_{ij}$ where each M_{ij} is P_i-coprimary and $\mu(P_i, M_{ij}) = 1$. Q.E.D.

We need one final preparatory lemma in order to prove Theorem 1.12.

(4.16) Lemma. *Let R be an E-ring and let M be a module which satisfies* *condition* $(*)$. *Then* $\hat{M}$ *satisfies condition* $(*)$ *as an* $\hat{R}$*-module.*

PROOF. Since depth $M = 1$, depth $\hat{M} = 1$. Thus, the problem is to show that if Q is a prime of $\hat{R}$ with $(\hat{R}/Q) = 1$, then $\mu(Q,\hat{M}) \leqslant 1$. By [21, Proposition 15, p. IV-25], $\mathrm{Ass}_{\hat{R}}(\hat{M}) = \bigcup_{i=1}^{h} \mathrm{Ass}_{\hat{R}}(R/P_i)\hat{}$. Since R is an E-ring, if $\dim(R/P_i) \geqslant 2$, no prime in $\mathrm{Ass}_{\hat{R}}(R/P)\hat{}$ has coheight 1. Thus, we need only show that if $P \in \mathrm{Ass}(M)$ and coheight $P = 1$, then if $Q \in \mathrm{Ass}_{\hat{R}}(R/P)\hat{}$ then $\mu(Q,\hat{M}) \leqslant 1$. But if $Q \in \mathrm{Ass}_{\hat{R}}(R/P)\hat{}$ then $Q \supset P\hat{R}$, i.e., there is a surjection $(R/P)\hat{} \twoheadrightarrow \hat{R}/Q$, which in turn yields an injection

$$\mathrm{Hom}_{\hat{R}}(\hat{R}/Q,\hat{M}) \hookrightarrow \mathrm{Hom}_{\hat{R}}((R/P)\hat{}\,,\hat{M}) \cong (\mathrm{Hom}_R(R/P,M))\hat{} \cong (\mathrm{Ann}_M P)\hat{}\,.$$

Since $\dim R/P = 1$, $\mathrm{Ann}_M P$ is a torsion-free (R/P)-module: otherwise, $m \in \mathrm{Ass}(\mathrm{Ann}_M P) \subset \mathrm{Ass}(M)$, a contradiction. Since $\mu(P,M) \leqslant 1$, rank $\mathrm{Ann}_M P \leqslant 1$, and hence $\mathrm{Ann}_M P$ is embeddable in R/P. Thus, we have $(\mathrm{Ann}_M P)\hat{} \hookrightarrow (R/P)\hat{}$ and so $\mathrm{Hom}_{\hat{R}}(\hat{R}/Q,\hat{M}) \hookrightarrow (R/P)\hat{}$. Since R is an E-ring if we localize $(R/P)\hat{}$ at Q it becomes a zero-dimensional Gorenstein ring, and we have an injection:

$$\mathrm{Hom}_{\hat{R}_Q}(\hat{R}_Q/Q\hat{R}_Q,\hat{M}_Q) \hookrightarrow ((R/P)\hat{}\,)_Q.$$

It follows that $\mu(Q,\hat{M}) = \mu(QR_Q,\hat{M}_Q) \leqslant \mu(Q,(R/P)\hat{}_Q) = 1$. Q.E.D.

We are now ready to prove the hardest part: that $(*)$ is sufficient for having S.C.I. when R is an E-ring. Of course, we have already done a good deal of the work.

(4.17) THEOREM. *Let* R *be an* E*-ring and let* M *be an* R*-module with* $\dim M > 0$. *Suppose that* M *satisfies condition* $(*)$. *Then* M *has small cofinite irreducibles.*

PROOF. We proceed by induction on $n = \dim M = \dim(R/\mathrm{Ann}\, M)$. By (4.16) we may assume that R is complete. By the induction hypothesis we know the result for all modules M' with $\dim M' < n$, even if the base ring is an E-ring and not necessarily complete.

We first consider the case in which $\dim M = 1$. Then all the primes in $\mathrm{Ass}(M)$ have coheight one. Since R is complete, we can map a complete regular local ring onto it. Then, by (4.1c), we may assume that R is complete regular. Let $I = \mathrm{Ann}(M)$. Then I will have pure height $(\dim R) - 1$, and hence, since R is regular and, in particular, Cohen-Macaulay, I will contain an R-sequence of length $(\dim R) - 1$. But R regular $\Rightarrow$ R Gorenstein, and so we may divide out by this R-sequence ((4.1c) again) and replace R by a one-dimensional Gorenstein ring.

Thus, if $\dim M = 1$ we may assume that $\dim R = 1$ and that R is a complete Gorenstein local ring. We have that for each minimal prime P of R,

$\mu(P, M) \leqslant 1$. But then M can be embedded in R. To see this, note that since depth $M = 1 = \dim R$, M is a Cohen-Macaulay module and so every nonzerodivisor in R is a nonzerodivisor on M. Let T be the multiplicative system of nonzerodivisors of R. Then $R \hookrightarrow T^{-1}R$, $M \hookrightarrow T^{-1}M$, and, as usual, it will suffice to embed $T^{-1}M \hookrightarrow T^{-1}R$. But $T^{-1}R$ is a product of zero-dimensional Gorenstein rings and $T^{-1}M \cong \prod_P M_P$ where P runs through the minimal primes of R. $\mu(P, M) \leqslant 1 \mu(PR_P, M_P) \leqslant 1 \Rightarrow M_P = 0$ or else M_P is an essential extension of $\kappa(P) = R_P/PR_P$. Since R_P is Gorenstein, it is the injective hull (maximal essential extension) of $\kappa(P)$, so that for each P we have an embedding $M_P \hookrightarrow R_P$ and hence an embedding $T^{-1}M \cong \prod_P M_P \hookrightarrow \prod_P R_P = T^{-1}R$, as required.

Since R is Gorenstein, it is approximately Gorenstein and has S.C.I. and, hence, so does its submodule M.

This completes the proof for the case $n = 1$.

We now suppose $n > 1$. We are retaining the assumption that R is complete and that the result holds for any R', M' if R' is an E-ring and $\dim M' < n$. Let $\mathrm{Ass}(M) = \{P_1, \ldots, P_h\}$ and embed $M \hookrightarrow \sum \oplus_{1 \leqslant i \leqslant h; 1 \leqslant j \leqslant \nu_i} M_{ij}$ precisely as in Proposition (4.15): M_{ij} is P_i-coprimary, $\nu_i = \mu(P_i, M)$, and $\mu(P_i, M_{ij}) = 1$, $1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant \nu_i$.

Then we can select $\sum_{i=1}^{h} \nu_i$ *distinct* primes P_{ij} with the following properties:

(1) For each i, j, $\dim R/P_{ij} = 1$.

(2) If $\dim(R/P_i) = 1$ (this implies $\nu_i = 1$, since M satisfies $(*)$), then $P_{ij} = P_i$.

(3) For each i, j, $P_{ij} \supset P_i$.

Note that in order to select these primes we must have $m \notin \mathrm{Ass}(M)$, and we also must have $\mu(P_i, M) = \nu_i = 1$ if $\mathrm{ht}(R/P_i) = 1$. (To select the P_{ij} first pick $P_{ij} = P_i$ for those i such that $\dim R/P_i = 1$. The other P_{ij} may be selected one at a time: given i, j, where $\dim R/P_i \geqslant 2$, let P_{ij} be any prime which contains P_i, has coheight 1 in R, and which has not already been used. This is possible, since R/P_i has infinitely many primes of coheight 1.)

By (4.3), we may replace M by $\sum \oplus_{ij} M_{ij}$: it will suffice to show that $M = \sum \oplus_{ij} M_{ij}$ has S.C.I.

Let an integer $N > 0$ be given. By (4.2) it will suffice to show that there is a module $M_N \subset m^N M$ such that M/M_N has S.C.I. Let M_{ij}^* be the localization of M_{ij} at P_{ij}. Since M_{ij} is P_i-coprimary and $P_i \subset P_{ij}$, $M_{ij} \subset M_{ij}^*$. Let $M_{ijk} = P_{ij}^k M_{ij}^* \cap M_{ij}$. Since $\bigcap_k P_{ij}^k M_{ij}^* = 0$, $\bigcap_k M_{ijk} = 0$. Since M_{ij} is a module over a *complete* local ring, and M_{ijk} is a decreasing sequence of submodules whose intersection is 0, we can choose an integer $k = N_{ij}$ such that $M_{ijN_{ij}} \subset m^N M_{ij}$.

Now, since R is complete, if R_{ij} denotes the localization of R at P_{ij}, then R_{ij} is excellent and, in particular, an E-ring. Since M_{ij} is P_i-coprimary and $\mu(P_i, M_{ij}) = 1$, we have that M_{ij}^* is $P_i R_{ij}$-coprimary and $\mu(P_i R_{ij}, M_{ij}^*) = 1$.

Hence, M_{ij}^* satisfies $(*)$ over R_{ij}. But $\dim M_{ij}^* < \dim M_{ij} \leqslant \dim M$; hence, M_{ij}^* has S.C.I. as an R_{ij}-submodule. It follows that we can choose an R_{ij}-submodule E_{ij}^* of M_{ij}^* which is cofinite, irreducible, and such that $E_{ij}^* \subset (P_{ij})^{N_{ij}} M_{ij}^*$. Let $E_{ij} = E_{ij}^* \cap M_{ij}$. By the choice of N_{ij}, $E_{ij} \subset m^N M_{ij}$. Moreover, it is easy to see that M_{ij}/E_{ij} is P_{ij}-coprimary and $\mu(P_{ij}, M_{ij}/E_{ij}) = 1$. Thus, we may let $M_N = \sum \oplus_{ij} E_{ij}$, for then $M_N \subset m^N M$ and $M/M_N = \sum \oplus_{ij}(M_{ij}/E_{ij})$ has S.C.I. by the case $n = 1$ (of course, we are using here the fact that the P_{ij} are all distinct). Q.E.D.

5. **Approximately Gorenstein rings, cyclic purity, and purity, revisited.** We have now done all the real work, and we can reap a harvest of corollaries.

(5.1) COROLLARY. *Let R be a Noetherian ring. The condition that R be approximately Gorenstein is local on the maximal ideals of R, i.e., R is approximately Gorenstein if and only if for every maximal ideal m of R, R_m is approximately Gorenstein.*

PROOF. This is immediate from the definition. Q.E.D.

(5.2) THEOREM. *Let R be a Noetherian ring. If R is an E-ring (in particular, if R is a locally excellent Noetherian ring, especially, a complete local ring), then R is approximately Gorenstein if and only if the following two conditions are satisfied:*

(1) *If m is a maximal ideal of R and $m \in \operatorname{Ass}(R)$, then R_m is a zero-dimensional Gorenstein ring.*

(2) *If $P \in \operatorname{Ass}(R)$ has coheight 1 in a maximal m of R, then $\mu(P, R) = 1$, i.e., $R/P \oplus R/P$ is not embeddable in R.*

Hence, for an arbitrary Noetherian ring R, R is approximately Gorenstein if and only if for each maximal ideal m of R, either R_m is a zero-dimensional Gorenstein ring or else $m \notin \operatorname{Ass}(R)$ and for each $Q \in \operatorname{Ass}(\hat{R}_m)$, if $\dim(\hat{R}_m/Q) = 1$ then $\mu(Q, \hat{R}_m) = 1$.

PROOF. The result is immediate from (5.1), (2.2c), (1.5), (1.11c), and (1.12). Q.E.D.

Of course, (1.6) is just the special case where R is complete local and $\dim R \geqslant 1$.

(5.3) PROOF OF (1.7). Since (1) $\Rightarrow$ (2), it suffices to show that either (2) or (3) implies the condition of (5.2). If (3) holds then the condition of (5.2) holds vacuously, because $\operatorname{Ass}(R)$ has no elements of coheight $\leqslant 1$ in a maximal ideal. (2) implies that all primes P in $\operatorname{Ass}(P)$ are minimal, and that for each $P \in \operatorname{Ass}(R)$, $\mu(P, R) = \mu(PR_P, R_P) = 1$, since R_P is Gorenstein. Q.E.D.

(5.4) EXAMPLE. Let r be an integer $\geqslant 2$. By Proposition 3.1 of [5], there is a local domain A of dimension 1 (thus, A is Cohen-Macaulay) such that $\hat{A}$ possesses a unique minimal prime P such that

(1) $P^2 = 0$,

(2) $P \cong (\hat{A}/P)^r$.

Thus, $\hat{A}$ is not generically Gorenstein, and, hence, neither is A. We can therefore embed $A \to S$, ideally closed, such that $A \to S$ is not pure. Since S is the direct limit of its finitely generated A-subalgebras, we may even take S to be finitely generated as an A-algebra. Moreover, A has equal characteristic 0. *Thus, there is a one-dimensional (Cohen-Macaulay) local domain A of equal characteristic* 0 *and a finitely generated extension algebra $A \to S$ such that $A \to S$ is ideally closed but not pure.*

The fact that A is not approximately Gorenstein is closely related to the fact that it has no canonical module; if A had a canonical module, it would be embeddable in it, and, hence, approximately Gorenstein. See (4.8b).

We conclude this section with two more corollaries of our main results.

(5.5) PROPOSITION. *Let R be an approximately Gorenstein Noetherian ring, for example, a reduced excellent Noetherian ring, and let S be a module-finite extension algebra. Suppose that $IS \cap R = I$ for every ideal I of R. Then R is a direct summand of S as an R-module.*

PROOF. The hypothesis guarantees that R is cyclically pure in S, and, since R is approximately Gorenstein, that R is pure in S. But then, since S/R is a finitely generated ($\Rightarrow$ finitely presented) R-module, R is a direct summand of S. Q.E.D.

(5.6) PROPOSITION. *Let R be any Noetherian ring and let x, y be analytic indeterminates over R. Then $R[x,y]$ and $R[[x,y]]$ are approximately Gorenstein.*

Moreover, if $R \hookrightarrow U$ is any algebra extension (or even an R-module extension) the following five conditions are equivalent:

(i) *R is pure in U.*

(ii) *$R[x,y]$ is cyclically pure in $U[x,y]$.*

(iii) *$R[[x,y]]$ is cyclically pure in $U[[x,y]]$.*

(iv) *$R[x,y]$ is pure in $U[x,y]$.*

(v) *$R[[x,y]]$ is pure in $U[[x,y]]$.*

PROOF. Let Q be a maximal ideal in $S = R[x,y]$ (respectively, $S = R[[x,y]]$) lying over say, P, in R. We must show that S_Q is approximately Gorenstein for each Q, or, equivalently, that $\hat{S}_Q$ is approximately Gorenstein for each Q. (Note that in the case $S = R[[x,y]]$, P must be maximal in R and we must have $Q = (P,x,y)S$.) Let $T = \hat{R}_P[x,y]$ (resp., $\hat{R}_P[[x,y]]$) and let $m = QT$. Then $\hat{T}_m = \hat{S}_Q$, so that it suffices to show that if A (let $A = \hat{R}_P$) is a complete local ring, then $B = A[x,y]$ (resp. $B = A[[x,y]]$) is approximately Gorenstein. But $\operatorname{Ass}_B B = \{pB \colon p \in \operatorname{Ass}_A(A)\}$ and if $p \in \operatorname{Ass} A$, $B/pB \cong (A/p)[x,y]$ (resp. $(A/p)[[x,y]]$) has the property that every maximal ideal has height at least two. Since B is excellent, by Theorem (1.7) it is approximately Gorenstein.

We now prove the equivalence of (i)–(v). From our point of view, the interest is in the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). In fact, (i) $\Rightarrow$ (iv) is trivial (since $R[x,y] \to S[x,y]$ arises by tensoring $R \to S$ with $R[x,y]$ over R), and (i) $\Rightarrow$ (v) is known (see [2]). Moreover, (v) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii) are quite evident.

To see why (ii) (or (iii)) implies (i) let $S = R[x,y]$ (resp., $R[[x,y]]$) and $T = U[x,y]$ (resp., $U[[x,y]]$). Since S is approximately Gorenstein, the cyclic purity of $S \to T$ implies the purity of $S \to T$. Now $R \to S$ is pure (in fact, R is an R-algebra retract of S via a homomorphism which kills $(x,y)S$), and so $R \to T$ is pure. Since U is an R-algebra retract of T, $R \to U$ is pure as well. Q.E.D.

(5.7) REMARK. It is well known (see [2], [4]), that if $R[x_1, \ldots, x_n] \to S[x_1, \ldots, x_n]$ is cyclically pure for every n, then R is pure in S, even if R is *not* Noetherian. The interesting point in (ii) $\Rightarrow$ (i) is that if R is Noetherian, we need only let $n = 2$.

6. Adjunction of analytic indeterminates fails to preserve purity in the non-Noetherian case. In [2] it is shown that if R is Noetherian and $R \hookrightarrow S$ is pure, then $R[[t]] \hookrightarrow S[[t]]$ is pure. In this section we construct a counterexample when R is not Noetherian. In fact, we prove the following:

(6.1) PROPOSITION-EXAMPLE. *Let K be any commutative ring, e.g., a field. Then there exists a countably generated K-algebra R, and a countably generated ring extension $R \hookrightarrow S$ such that*

(1) *$R \hookrightarrow S$ is pure.*

(2) *$R[[t]] \to S[[t]]$ is not even cyclically pure: in fact there is a principal ideal of $R[[t]]$ which is not contracted from $S[[t]]$.*

PROOF. We construct R, S explicitly as follows. Let $\{X_j, Z_{ij}, W_j\colon i,j\}$, where i,j are nonnegative integers and $i \leqslant j$, be countably many indeterminates. Let $T = K[X_j, Z_{i,j}\colon i \leqslant j]$. In T, if $j', j \geqslant n \geqslant 0$, let

$$p(j,j',n) = \sum_{i=0}^{n} X_i(Z_{n-i,j} - Z_{n-i,j'}).$$

Let I be the ideal of T generated by the elements $p(j,j',n)$ and let $R = T/I$. We write x_i, z_{ij} for the images of X_i, Z_{ij}, respectively, in R. Thus,

$$R = K[x_i, z_{ij}\colon i \leqslant j]$$

and if we let $y_n = \sum_{i=0}^{n} x_i z_{n-i,j}$ for any $j \geqslant n$ the relations $p(j,j',n)$ tell us precisely that y_n is independent of the choice of $j \geqslant n$ for each n.

Let $U = R[W_j\colon j]$ and for each n let $p_n \in U$ be defined by

$$p_n = y_n - \sum_{i=0}^{n} x_i W_{n-i}.$$

Let J be the ideal of U generated by the p_n and let $S = U/J$.

We shall show that R is pure in S but that $R[[t]]$ is not cyclically pure in $S[[t]]$.

To see that R is pure in S let $S_n = R[W_0, \ldots, W_n]/(p_0, \ldots, p_n)$. Clearly, in an obvious way, $S = \text{proj lim}_n S_n$. Therefore, it suffices to show that $R \to S_n$ is pure for each n. But R is a direct summand of S_n as an R-module: in fact, R is an R-algebra retract. To see this, simply map $R[W_0, \ldots, W_n] \to R$ (as R-algebras) by sending W_i to z_{in}. Then p_i maps to $y_i - \sum_{i=0}^{n} x_i z_{n-i,n} = 0$, $0 \leqslant i \leqslant n$, and so we have the required R-algebra retraction

$$S_n = R[W_0, \ldots, W_n]/(p_0, \ldots, p_n) \to R.$$

It remains to show that $R[[t]] \to S[[t]]$ is not cyclically pure. We simply exhibit a principal ideal of $R[[t]]$ which is not contracted. Let

$$x = \sum_{i=0}^{\infty} x_i t^i \in R[[t]],$$

$$y = \sum_{i=0}^{\infty} y_i t^i \in R[[t]],$$

and

$$w = \sum_{i=0}^{\infty} w_i t^i \in S[[t]].$$

Then $y \in xS[[t]]$; in fact, the relations p_i are precisely what we need to guarantee that $y = wx$. To complete the proof, it will suffice to show that $y \notin xR[[t]]$, for then $xR[[t]]$ is a principal ideal of $R[[t]]$ which is not contracted from $S[[t]]$.

Suppose, to the contrary, that $y \in xR[[t]]$. We shall derive a contradiction. The fact that $y \in xR[[t]]$ simply means that there are elements $r_0, \ldots, r_i, \ldots \in R$ such that

$$y = x \sum_{i=0}^{\infty} r_i t^i,$$

or, in other words, that the r_i satisfy the countable system of equations

$$
\text{(E)} \qquad \begin{cases} y_0 = x_0 r_0, \\ y_1 = x_1 r_0 + x_0 r_1, \\ \quad \cdots \\ \quad \cdots \\ \quad \cdots \\ y_n = x_n r_0 + \cdots + x_0 r_n, \\ \quad \cdots \\ \quad \cdots \\ \quad \cdots \end{cases}
$$

Given a supposed solution, we can choose an integer m so large that $r_0 \in K[x_0, \ldots, x_m, z_{ij}, i \leqslant j \leqslant m] = R_m \subset R$. We shall obtain a contradiction by showing that if $r_0 \in R_m$ then even the two equations

$$
(\mathrm{E}_m) \qquad \begin{cases} y_m = r_0 x_m + \cdots + r_m x_0, \\ y_{m+1} = r_0 x_{m+1} + \cdots + r_{m+1} x_0, \end{cases}
$$

have no solution in R.

To this end, we introduce an auxiliary ring. Let U_0, U_1, V_0, V'_1, and Q be indeterminates over K, let $B = K[U_0, U_1, V_0, Q]$, and let $A = B/(U_0 Q)$. We denote the images of the variables in A by u_0, u_1, v_0, q. We define $v'_0 = v_0 + q$ and note that $u_0 v_0 = u_0 v'_0$.

Let ϕ be the unique K-homomorphism of $T = K[X_i, Z_{ij}\colon i \leqslant j]$ to A such that:

$$
\begin{aligned}
&\phi(X_i) = 0 \quad \text{if } i < m \text{ or } i > m+1, \\
&\phi(X_m) = u_0 \quad \text{and} \quad \phi(X_{m+1}) = u_1, \phi(Z_{ij}) = 0 \quad \text{if } i \geqslant 1, \\
&\phi(Z_{0j}) = v_0, \quad \text{if } j \leqslant m, \\
&\phi(Z_{0j}) = v'_0 \quad \text{if } j \geqslant m+1.
\end{aligned}
$$

It is easy to check that for all n, j, j', where $j, j' \geqslant n$, $\phi(p(j, j', n)) = 0$. Hence ϕ induces a K-homomorphism

$$
\psi\colon R \to A.
$$

Let a_i denote the image of r_i in A. Now, $\psi(R_m) = K[u_0, v_0] \subset A$ and so

$a_0 \in K[u_0, v_0]$. Moreover, $\psi(y_\nu) = \psi(\sum_{i=0}^{\nu} x_i z_{\nu-i,\nu})$, and so $\psi(y_\nu) = 0$ if $\nu < m$ while $\psi(y_m) = u_0 v_0$ and $\psi(y_{m+1}) = u_1 v_0'$. If we apply ψ to the equations (E_m) we obtain

(1) $u_0 v_0 = u_0 a_0$,

(2) $u_1 v_0' = u_0 a_1 + u_1 a_0$,

over A, and we know $a_0 \in K[u_0, v_0]$. Now, $U_0 QB \cap K[U_0, V_0] \subset QB \cap K[U_0, V_0] = 0$, so that $K[u_0, v_0] \cong K[U_0, V_0]$. But then $u_0 v_0 = u_0 a_0$ and $a_0 \in K[u_0, v_0] \Rightarrow a_0 = v_0$. Substituting in (2), we have $u_1 v_0' - u_1 v_0 \in u_0 A$, or, passing to B, that $U_1(V_0 + Q) - U_1 V_0 \in (U_0, U_0 Q)B = U_0 B$, whence $U_1 Q \in U_0 B$, a contradiction. Q.E.D.

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